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The Σ^2-Conjecture for Metabelian Groups and Some New Conjectures: The Split Extension Case

Dessislava H. Kochloukova

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INTRODUCTION

In this paper we discuss the Σ^2-Conjecture for metabelian groups and answer it positively in the split extension case. The Σ^2-Conjecture is a low dimensional case of the Σ^m-Conjecture that describes the higher geometric invariants $\{\Sigma^m(G)\}_{m\in\mathbb{N}}$ and $\{\Sigma^m(G,\mathbb{Z})\}_{m\in\mathbb{N}}$ of a finitely generated metabelian group G. In general $\Sigma^m(G,\mathbb{Z})$ is defined for every finitely generated group G and $\Sigma^m(G)$ for groups of type F_m. The higher geometric invariants were introduced in [7, 22] and generalize the geometric invariants defined in [6] and [8]. By definition they are subsets of the unit sphere $S(G) = (\operatorname{Hom}(G,\mathbb{R})\backslash\{0\}/\sim$, where $\sim$ is the equivalent relation on $\operatorname{Hom}(G,\mathbb{R})\backslash\{0\}$ with classes $[\chi] = \mathbb{R}_{>0}\chi$. If A is a (left) $\mathbb{Z}G$-module

$$\Sigma^m(G,A) = \{[\chi] \in S(G) | A \text{ is of type } FP_m \text{ over } \mathbb{Z}G_\chi\},$$

where $G_\chi = \{g \in G | \chi(g) \geq 0\}$. We do not state the definition of $\Sigma^m(G)$ but note it is a homotopical version of $\Sigma^m(G,\mathbb{Z})$. For general finitely generated groups $\Sigma^1(G)$ and $\Sigma^1(G,\mathbb{Z})$ coincide. As shown in [20] this is false in higher dimensions.

The importance of the higher geometric invariants of a group G lies in the fact that they control the homological and homotopical properties of the subgroups of G which contain the derived subgroup (see [7, 22]). More about the higher geometric invariants can be found in [12, 18, 19]. The following conjecture from R. Bieri suggests that the higher geometric invariants of a metabelian group G are determined by $\Sigma^1(G)$.

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THE Σ^m-CONJECTURE. *If G is a metabelian group of type FP_m then*

$$0 \notin \operatorname{conv}_{\leq m}\left(\mathbb{R}_{>0}\Sigma^1(G)^c\right)$$

and

$$\Sigma^m(G)^c = \Sigma^m(G,\mathbb{Z})^c = \left(\operatorname{conv}_{\leq m}\left(\mathbb{R}_{>0}\Sigma^1(G)^c\right)\right)/\sim ,$$

where upper index c denotes complement in $S(G)$; $\operatorname{conv}_{\leq m}X$ for $X \subseteq \mathbb{R}^n$ is the convex hull of not more than m-elements of X.

Thought the Σ^m-Conjecture is still an open problem some cases of it are answered positively in [14, 15, 18, 19]. It is related to the following conjecture suggested in [3].

THE FP_m-CONJECTURE. *A finitely generated metabelian group is of type FP_m if and only if*

$$0 \notin \operatorname{conv}_{\leq m}\left(\mathbb{R}_{>0}\Sigma^1(G)^c\right).$$

The FP_m-Conjecture has been proved in many cases (see [1, 3, 8, 10, 13, 21]). In [8] the case $m = 2$ is resolved; furthermore Bieri and Strebel show that for finitely generated metabelian groups the properties FP_2 and finite presentability are equivalent. Our proof of the Σ^2-Conjecture in the split extension case is inspired by the geometric methods introduced in [8]; the core of the proof is Theorem A1 stated below.

THEOREM A1. *Suppose G is a split extension of A by Q, A and Q are abelian, G is of type FP_2, and φ is a non-trivial real character of Q such that $\gamma(\varphi) \notin \operatorname{conv}_{\leq 2}(\mathbb{R}_{>0}\Sigma^1(G)^c)$, where $\gamma\colon \operatorname{Hom}(Q,\mathbb{R}) \to \operatorname{Hom}(G,\mathbb{R})$ is induced by the projection $G \to Q$. Then $[\gamma(\varphi)] \in \Sigma^2(G)$.*

It is easy to see that if χ is a non-trivial discrete character of a group G that is an extension of A by Q, where A and Q are abelian, G is of type FP_2, and A is not in the kernel of χ then $[\chi] \in \Sigma^2(G)$. Indeed in this case $\operatorname{Ker}\chi$ is an extension of $A \cap \operatorname{Ker}\chi$ by a group isomorphic to a subgroup of finite index in Q and by the characterization theorem for finitely presented metabelian groups in [8] $\operatorname{Ker}\chi$ is finitely presented. We consider the action of $\mathbb{Z}$ on $\mathbb{R}$ given by translations and extend it to an action of G given by the map $\chi\colon G \to \mathbb{Z}$. This action of G has the property that its edge stabilizers are finitely generated and the vertex stabilizers are finitely presented (in fact all edge and vertex stabilizers coincide with $\operatorname{Ker}\chi$). As the identity on $\mathbb{R}$ is a χ-equivariant height function (for the definition of equivariant height functions see [19]) we can apply [19, Theorem 4.1] and deduce $[\chi] \in \Sigma^2(G)$. In general this result holds not only for discrete characters. As shown in [17] if G is a finitely presented discrete group, N is a normal locally polycyclic-by-finite subgroup of G, and χ is a real character of G that does not vanish on N then $[\chi] \in \Sigma^2(G)$. A simple proof in the case when $G = N \rtimes Q$, N and Q are abelian, and G is finitely generated can be found in [15]. This together with Theorem A1 implies that $\Sigma^2(G)^c \subseteq (\mathrm{conv}_{\leq 2}(\mathbb{R}_{>0}\Sigma^1(G)^c))/\sim$ in the split extension case. The inverse inclusion is proved in [12] and holds for general finitely generated metabelian groups.

COROLLARY A2. *The Σ^2-Conjecture holds in the split extension case.*

In [16] we introduced the following conjecture that classifies the homological type of some special modules over finitely generated metabelian groups.

CONJECTURE 1. *Suppose $A \to G \to Q$ is a short exact sequence of groups, A and Q are abelian, G is finitely generated, and B is a finitely generated module over $\mathbb{Z}Q$ that is viewed as an $\mathbb{Z}G$-module via the projection $G \to Q$. Then B is of type FP_m over $\mathbb{Z}G$ if and only if*

$$0 \notin \mathbb{R}_{>0}\Sigma^0(G,B)^c + \mathrm{conv}_{\leq m}\left(\mathbb{R}_{>0}\Sigma^1(G)^c\right).$$

Some cases of Conjecture 1 are proved in [16]. There we showed that the conjecture holds in dimension one in the split extension case and that the forward implication holds if both A and B have prime characteristic $p > 0$. In this paper we suggest a description of the higher geometric invariants of the modules considered in Conjecture 1.

CONJECTURE 2. *In conditions of Conjecture* 1 *if B is of type FP_m over $\mathbb{Z}G$*

$$\Sigma^m(G,B)^c = \left(\mathbb{R}_{>0} \cdot \Sigma^0(G,B)^c + \mathrm{conv}_{\leq m}\left(\mathbb{R}_{>0}\Sigma^1(G)^c\right)\right)/\sim \,.$$

The following theorem can serve as a motivation of Conjecture 2.

THEOREM B. *Suppose $A \to G \to Q$ is a split short exact sequence of groups, A and Q are abelian, G is finitely generated, and B is a finitely generated module over $\mathbb{Z}Q$ that is of type FP_1 over $\mathbb{Z}G$, i.e., finitely presented. Then*

$$\Sigma^1(G,B)^c = \left(\mathbb{R}_{>0}\Sigma^1(G)^c + \mathbb{R}_{>0}\Sigma^0(G,B)^c\right)/\sim \,.$$

In [15] we suggested a criterion for the discrete characters of $\Sigma^m(G,\mathbb{Z})$ in terms of the homological properties of A. There we showed a proof in a low dimensional split extension case without referring to Conjecture 2. Now we state this criterion for non-discrete characters and note that it would be a straightforward corollary of Conjecture 2 and the Σ^m-Conjecture once both conjectures are proved.

CONJECTURE 3. *If $A \to G \to Q$ is a short exact sequence of groups, A and Q are abelian, and G is of type FP_m then*

$$\Sigma^m(G, \mathbb{Z}) = \Sigma^{m-1}(G, A).$$

1. PRELIMINARIES

Let G be a split extension of A by Q, where A and Q are abelian groups and G is finitely generated, and we view A as a left $\mathbb{Z}Q$-module via conjugation. As G is finitely generated there exists a finite set $a_1, \dots, a_s$ generating A over $\mathbb{Z}Q$.

We note that the Σ^2-Conjecture holds for G if it holds for a subgroup H of finite index in G. Indeed by [9, Corollary B4.13] if χ is a non-zero real character of G then $[\chi] \in \Sigma^m(G, \mathbb{Z})$ if and only if $[\chi|_H] \in \Sigma^m(H, \mathbb{Z})$. The homotopical version of this result is given in [19, 2.7 Corollary]. Then without loss of generality we can assume $Q \simeq \mathbb{Z}^n \subset \mathbb{R}^n$ and consider Q equipped with the standard inner product inherited from $\mathbb{R}^n$.

We change the notations a bit to be consistent with the ones used in [8] and denote $\Sigma_A(Q) = \Sigma^0(Q, A), \Sigma_A^c(Q) = S(Q) \setminus \Sigma_A(Q)$. Then the map $\mathrm{Hom}(Q, \mathbb{R}) \to \mathrm{Hom}(A \rtimes Q, \mathbb{R})$ induced by the projection $A \rtimes Q \to Q$ gives a bijection between $\Sigma_A^c(Q)$ and $\Sigma^1(A \rtimes Q)^c$. We assume A is 2-tame as a $\mathbb{Z}Q$-module, i.e., $\Sigma_A^c(Q) \cap -\Sigma_A^c(Q) = \emptyset$ and fix a character $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R}) \setminus \{0\}$, $\varphi \notin \mathrm{conv}_{\leq 2}(\mathbb{R}_{>0} \cdot \Sigma_A^c(Q))$. By [8, Proposition 2.1] $A = \mathbb{Z}Q_\varphi a_1 + \cdots + \mathbb{Z}Q_\varphi a_s$.

DEFINITION. Let F be the free group on the set $Y = \{{}^q b_i | q \in Q_\varphi, 1 \leq i \leq s\}$. We view Y as a left Q_φ-set via ${}^{q_1}({}^{q_2}b_i) := {}^{(q_1 q_2)} b_i$ and define μ: $F \to A$ to be the group homomorphism sending ${}^q b_i$ to ${}^q a_i$ for all $q \in Q_\varphi$, $1 \leq i \leq s$.

One of the main results in this paper is to show that $\mathrm{Ker}\,\mu$ is generated by finitely many Q_φ-orbits as a normal subgroup of F (see Corollary 3.5). The proof of this result is rather long and the following lemma is its starting point.

LEMMA 1.1. *There exists a normal subgroup N of F such that $N \subset \mathrm{Ker}\,\mu$, N is Q_φ-invariant, N is generated as a normal subgroup of F by finitely many orbits under the action of Q_φ, and furthermore we can define an action of Q on F/N which via μ corresponds to the action of Q on A via conjugation.*

Proof. By [8, Proposition 2.1] there exists an element λ_0 in the centralizer $C_{\mathbb{Z}Q}(A)$ of A in $\mathbb{Z}Q$ such that for every $q \in \operatorname{supp} \lambda_0$, $\varphi(q) > 0$. Furthermore let q_0 be an element of $Q_\varphi \setminus \mathrm{Ker}\,\varphi$ such that $q_0^{-1} q \in Q_\varphi$ for all $q \in \operatorname{supp} \lambda_0$. We note that $Q = \bigcup_{z<0} q_0^z Q_\varphi$.

Let $\{q_1, \dots, q_t\}$ be the support of λ_0, so $\lambda_0 = \sum_{1 \le i \le t} z_i q_i$ for some $z_i \in \mathbb{Z} \setminus \{0\}$. We define N to be the normal subgroup of F generated by $\{{}^{Q_\varphi}[b_i(({}^{z_1 q_1}b_i)({}^{z_2 q_2}b_i) \cdots ({}^{z_t q_t}b_i))^{-1}] | 1 \le i \le s\} \subset \operatorname{Ker} \mu$. Here ${}^{z_j q_j}b_i$ denotes the z_jth power of ${}^{q_j}b_i$.

For $q \in Q_\varphi$ we define ${}^q c_i$ to be the image of ${}^q b_i$ in F/N and note that since N is Q_φ-invariant, F/N inherits from F a (left) action of Q_φ. Now we define a group homomorphism $\tilde{\mu}$: $F/N \to F/N$ that sends c_i to $({}^{z_1 q_0^{-1} q_1}c_i)({}^{z_2 q_0^{-1} q_2}c_i) \cdots ({}^{z_t q_0^{-1} q_t}c_i)$ and $\tilde{\mu}({}^q c_i) = {}^q\tilde{\mu}(c_i)$ for $q \in Q_\varphi$. Note that ${}^{q_j}\tilde{\mu}(c_i) = {}^{q_0^{-1} q_j}[({}^{z_1 q_1}c_i)({}^{z_2 q_2}c_i) \cdots ({}^{z_t q_t}c_i)] = {}^{q_0^{-1} q_j}c_i$. Then

$$\left({}^{z_1 q_1}\tilde{\mu}(c_i)\right)\left({}^{z_2 q_2}\tilde{\mu}(c_i)\right) \cdots \left({}^{z_t q_t}\tilde{\mu}(c_i)\right) = \left({}^{z_1 q_0^{-1} q_1}c_i\right)\left({}^{z_2 q_0^{-1} q_2}c_i\right) \cdots \left({}^{z_t q_0^{-1} q_t}c_i\right)$$
$$= \tilde{\mu}(c_i)$$

and so $\tilde{\mu}$ is well defined.

Now we define an action of Q on F/N as follows. The monoid Q_φ acts as before; the action of q_0^{-k}, $k \in \mathbb{N}$, is given by the homomorphism $\tilde{\mu}^k$. It remains only to note that the action is well defined; i.e., $\tilde{\mu}({}^{q_0}c_i) = {}^{q_0}(\tilde{\mu}(c_i)) = ({}^{z_1 q_1}c_i)({}^{z_2 q_2}c_i) \cdots ({}^{z_t q_t}c_i) = c_i$ in F/N. This completes the proof of the lemma.

DEFINITION. From now on we write ${}^q c$, where $c \in F/N$, $q \in Q$, for the image of c under the action of q.

The element q_0 that was used in the definition of the action of Q on F/N will appear several times throughout the next sections. The following simple lemma gives more information about the action of q_0^{-1}.

LEMMA 1.2. *Let* $\prod_{q \in Q_\varphi, r_q \in \mathbb{Z}} ({}^{r_q q}c_i) = \alpha$ *be an element of* F/N *and* m *be a positive integer. Then* ${}^{q_0^{-m}}\alpha$ *belongs to the subgroup of* F/N *generated by* $\{{}^{Q_\varphi}\alpha\}$ *and all* F/N*-conjugates of* $\{{}^{Q_\varphi}[{}^{q'}c_i, {}^{q''}c_i] | q', q'' \in Q_\varphi, \max\{|q''|, |q'|\} \le s_0 m + \max_{r_q \ne 0}|q|\}$, $s_0 = \max_{q_i \in \operatorname{supp} \lambda_0}|q_0^{-1} q_i|$.

Proof. We induct on m. Let $m = 1$, then

$${}^{q_0^{-1}}\alpha = \prod_{q \in Q_\varphi} \left({}^{r_q q q_0^{-1}}c_i\right) = \prod_{q \in Q_\varphi} \left[\left({}^{z_1 q q_0^{-1} q_1}c_i\right)\left({}^{z_2 q q_0^{-1} q_2}c_i\right) \cdots \left({}^{z_t q q_0^{-1} q_t}c_i\right)\right]^{r_q}$$
$$\equiv \left({}^{z_1 q_0^{-1} q_1}\alpha\right)\left({}^{z_2 q_0^{-1} q_2}\alpha\right) \cdots \left({}^{z_t q_0^{-1} q_t}\alpha\right)$$

modulo the normal subgroup of F/N generated by the elements $\{[{}^{q'}c_i, {}^{q''}c_i] | q', q'' \in Q_\varphi, \max\{|q''|, |q'|\} \le s_0 + \max_{r_q \ne 0}|q|\}$. We note that $q_0^{-1} q_i \in Q_\varphi$ for all $1 \le i \le t$.

The inductive step follows immediately from the inductive hypothesis and the fact that for $q', q'' \in Q_\varphi$, ${}^{q_0^{-1}}[{}^{q'}c_i, {}^{q''}c_i]$ belongs to the normal subgroup of F/N generated by $\{[{}^{h'}c_i, {}^{h''}c_i] | h', h'' \in Q_\varphi, |h'| \leq |q'| + s_0, |h''| \leq |q''| + s_0\}$. Indeed

$$\begin{aligned} {}^{q_0^{-1}}\left[{}^{q'}c_i, {}^{q''}c_i\right] &= \left[{}^{q'q_0^{-1}}c_i, {}^{q''q_0^{-1}}c_i\right] \\ &= \left[\left({}^{z_1 q' q_0^{-1} q_1}c_i\right)\left({}^{z_2 q' q_0^{-1} q_2}c_i\right) \cdots \left({}^{z_t q' q_0^{-1} q_t}c_i\right),\right. \\ &\qquad \left.\left({}^{z_1 q'' q_0^{-1} q_1}c_i\right)\left({}^{z_2 q'' q_0^{-1} q_2}c_i\right) \cdots \left({}^{z_t q'' q_0^{-1} q_t}c_i\right)\right] \end{aligned}$$

belongs to the normal subgroup of F/N generated by $\{[{}^{q'q_0^{-1}q_k}c_i, {}^{q''q_0^{-1}q_j}c_i] | 1 \leq k, j \leq t\}$ and $|q'q_0^{-1}q_j| \leq |q'| + |q_0^{-1}q_j| \leq |q'| + s_0$, $|q''q_0^{-1}q_j| \leq |q''| + s_0$ for all $1 \leq j \leq t$.

Now we state several geometric lemmas about the invariant Σ that will play a vital role in the proof of Corollary 3.5.

LEMMA 1.3. *If $\Sigma_A^c(Q) \cap -\Sigma_A^c(Q) = \emptyset$ and $\{x_1, x_2\} \subseteq -\Sigma_A^c(Q)$ then the angle between x_1 and x_2, considered as vectors in $\mathbb{R}^n$ with beginning at the origin, cannot be arbitrary close to π.*

Proof. As shown in [4] Σ_A^c is a rationally defined spherical polyhedron; i.e., it is a union of finitely many finite intersections of closed half spheres. Lemma 1.3 is an obvious corollary of the polyhedral structure of Σ_A^c.

LEMMA 1.4. *Suppose $\varphi \notin \mathrm{conv}_{\leq 2}(\mathbb{R}_{>0} \cdot \Sigma_A^c(Q))$ and write $[\varphi]$ for the projection of φ to $S(Q)$. Then there exists a positive real number $\alpha \in (0, \pi)$ such that for every set $X = \{x_1, x_2, x_3\}$ with $x_1, x_2 \in -\Sigma_A^c(Q)$, $x_3 \in \mathbb{R}_{>0}[\varphi]$ there exists $u \in \mathbb{R}^n$ such that $\angle\{x, u\} < \frac{\pi}{2} - \frac{\alpha}{2}$ for all $x \in X$.*

Proof. The lemma follows from the structure of $\Sigma_A^c(Q)$ as a rationally defined spherical polyhedron.

Later on we will need the following geometrical results from R. Bieri, J. R. J. Groves, and J. Harlander.

PROPOSITION 1.5. *There exist a finite subset $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{m_0}\}$ of the centralizer $C_{\mathbb{Z}Q}(A)$ of A in $\mathbb{Z}Q$ and a positive real number ν_0 such that for every $[\chi] \in \Sigma_A(Q)$ there is $\lambda_i \in \Lambda$ with the property that $\chi(q) > \nu_0$ for all $q \in \mathrm{supp}\, \lambda_i$.*

Proof. The proposition is a special case of the main theorem of [5]. A weaker version of it can be found in [11, Lemma 2.5(1)].

PROPOSITION 1.6. *There exists a positive real number ρ_0 such that if $x \in \mathbb{R}^n$ with $|x| > \rho_0$, $\frac{x}{|x|} \in -\Sigma_A(Q)$ then there is an element $\lambda \in \Lambda$ such that for every $q \in \mathrm{supp}\, \lambda$, $|x + q| < |x|$.*

Proof. The result is a stronger version of [11, Lemma 3.7(1)] and can be deduced by a slight modification of the original argument in [11]; i.e., we have to use Proposition 1.5 instead of [11, Lemma 2.5(1)] in the proof of [11, Lemma 3.7(1)].

LEMMA 1.7. *Let α be a real number in the interval $(0, \pi)$ and n be a positive integer. Then there exist positive real numbers $\rho_1(\alpha, n)$ and $\rho_2(\alpha, n)$, depending on α and n, satisfying the following.*

Let X be a finite set in $\mathbb{R}^n$ and let $u \in \mathbb{R}^n$ be of length 1. Suppose that $\angle\{x, u\} < \frac{\pi}{2} - \frac{\alpha}{2}$ and $|x| > \rho_1(\alpha, n)$ for each $x \in X$. Then there exists $v \in \mathbb{Z}^n$ with $|v| < \rho_2(\alpha, n)$ and $|v + x| < |x|$ for each $x \in X$.

Proof. This is [11, Lemma 3.8].

2. THE DEFINITION AND SOME PROPERTIES OF $X(m)$

From now on we consider φ as a real character or $\mathbb{R}^n$ given by the obvious $\mathbb{R}$-linear extension of the original character φ of $Q = \mathbb{Z}^n$.

DEFINITION. Let $O_r(y)$ denote the closed ball with center y and radius r in $\mathbb{R}^n$ and r_1, r_2 be positive real numbers such that $2r_1 < r_2$. Then we define D_{r_1, r_2} to be the union of all balls $O_{r_1}(y)$, where $y \in O_{r_2 - r_1}(0)$ and $\varphi(y) \geq 0$. We set $X(m) = D_{\sqrt{m}, m}$ for $m \geq 5$, $m \in \mathbb{N}$.

DEFINITION. Let $\Lambda = \{\lambda_1, \dots, \lambda_{m_0}\}$ be the set given by Proposition 1.5 and $\{q_{j,1}, q_{j,2}, \dots, q_{j,t_j}\}$ be the support of λ_j, so $\lambda_j = \sum_k z_{j,k} q_{j,k}$ for some $z_{j,k} \in \mathbb{Z} \setminus \{0\}$. We define some special elements $c_{i,j} \in F/N$ by

$$c_{i,j} = c_i\left[\left({}^{z_{j,1} q_{j,1}} c_i\right)\left({}^{z_{j,2} q_{j,2}} c_i\right) \cdots \left({}^{z_{j,t_j} q_{j,t_j}} c_i\right)\right]^{-1}.$$

LEMMA 2.1. *There exists a positive integer ρ such that for $m \geq \rho$ and $q \in X(m) \cap Q$ the element ${}^q c_{i,j}$ belongs to the subgroup of F/N generated by $\{{}^{Q_\varphi} c_{i,j}\}$ and all F/N-conjugates of $\{{}^{Q_\varphi}[{}^{q'} c_i, {}^{q''} c_i] | q', q'' \in X(m-1) \cap Q_\varphi\}$.*

Proof. Note that there exists a positive real number δ such that for all $m \geq 5$ $(X(m) \cap Q) \setminus Q_\varphi \subseteq q_0^{-\delta[\sqrt{m}]} Q_\varphi$. For sufficiently big m we have $s_0[\sqrt{m}]\delta + \max_{1 \leq i \leq m_0,\, 1 \leq k \leq t_i} |q_{i,k}| \leq m - 1$, where $[\sqrt{m}]$ denotes the integral part of $\sqrt{m}$. Then the lemma follows immediately from Lemma 1.2.

From now on we fix ρ and ρ_0 to be positive integers satisfying the conclusions of Lemma 2.1 and Proposition 1.6, respectively.

LEMMA 2.2. *Suppose $m \geq \rho$, $q' \in O_{\sqrt{m}}(y') \subset X(m)$, $q' \in Q$, $y' \in \mathbb{R}^n$, and $|q' - y'| > \rho_0$, $(q' - y')/|q' - y'| \in -\Sigma_A(Q)$. Then ${}^{q'} c_i$ belongs to the* *subgroup of F/N generated by* $\{{}^{q}c_i \mid q \in Q, |q - y'| < |q' - y'|\} \cup \{{}^{Q_\varphi}c_{i,j} \mid 1 \le j \le m_0\}$ *and all F/N-conjugates of* $\{{}^{Q_\varphi}[{}^{h'}c_i, {}^{h''}c_i] \mid h', h'' \in X(m-1) \cap Q\}$.

Proof. By Proposition 1.6 there exists an element $\lambda_j \in \Lambda$ such that for all $q_{j,k} \in \operatorname{supp} \lambda_j$ we have $|q' - y' + q_{j,k}| < |q' - y'|$. Now we consider the element $c_{i,j}$ and note that

$${}^{q'}c_i = {}^{q'}c_{i,j}\left({}^{z_{j,1}q'q_{j,1}}c_i\right)\left({}^{z_{j,2}q'q_{j,2}}c_i\right) \cdots \left({}^{z_{j,t_j}q'q_{j,t_j}}c_i\right).$$

The proof is now completed by Lemma 2.1.

3. MORE GEOMETRIC LEMMAS

DEFINITION. If $\{q_i\}_{1 \le i \le 4}$ are elements of $\mathbb{R}^n$, we write $\{q_1, q_2\} \prec \{q_3, q_4\}$ if and only if $\max\{|q_1|, |q_2|\} \le \max\{|q_3|, |q_4|\}$ and if equality occurs then $\min\{|q_1|, |q_2|\} < \min\{|q_3|, |q_4|\}$.

LEMMA 3.1. *Suppose δ is a real positive number. Then there exists a positive real number $\mu_1(\delta)$, depending on δ, such that if $m \ge \mu_1(\delta)$, $q', q'' \in Q$, $y', y'' \in \mathbb{R}^n$, $q' \in O_{\sqrt{m}}(y') \subset X(m)$, $q'' \in O_{\sqrt{m}}(y'') \subset X(m)$, and $|q' - y'| \le \delta$, $q'' \notin X(m-1)$, $(q'' - y'')/|q'' - y''| \in -\Sigma_A^c(Q)$ then there exists $v \in -Q_\varphi$ such that $\{v + q' - y', v + q'' - y''\} \prec \{q' - y', q'' - y''\}$.*

Proof. Since $q'' \notin X(m-1)$ we deduce that the distance $d(q'', \{x \in \mathbb{R}^n \mid \varphi(x) \ge 0,\ |x| \le m - 1 - \sqrt{m-1}\}) > \sqrt{m-1}$. At the same time $d(q'', \{x \in \mathbb{R}^n \mid \varphi(x) \ge 0,\ |x| \le m - 1 - \sqrt{m-1}\}) \le |q'' - y''| + d(y'', \{x \in \mathbb{R}^n \mid \varphi(x) \ge 0,\ |x| \le m - 1 - \sqrt{m-1}\}) \le |q'' - y''| + m - \sqrt{m} - (m - 1 - \sqrt{m-1})$. Hence $|q'' - y''| > \sqrt{m-1} - (1 - \sqrt{m} + \sqrt{m-1}) = \sqrt{m} - 1$.

Since $(q'' - y'')/|q'' - y''| \in -\Sigma_A^c(Q)$ by Lemma 1.4, for $x_1 = x_2 = (q'' - y'')/|q'' - y''|$ the angle between $[\varphi]$ and $(q'' - y'')/|q'' - y''|$ is less than $\pi - \alpha$. We want to apply Lemma 1.7 to the set $X = \{z[\varphi], q'' - y''\}$, where z is a sufficiently big positive integer and $[\varphi] \in S(Q)$ is viewed as an element of $\mathbb{R}^n$ via the isomorphism $\mathbb{R}^n \simeq \operatorname{Hom}(Q, \mathbb{R})$ induced by $Q \simeq \mathbb{Z}^n$. Indeed $|q'' - y''| \ge \sqrt{m} - 1 > \rho_1(\alpha, n)$ for sufficiently big m and we can take $z > \rho_1(\alpha, n)$. Then by Lemma 1.7 there exists $v \in Q$ such that $|v| \le \rho_2(\alpha, n)$, $|v + q'' - y''| < |q'' - y''|$, $|v + z[\varphi]| < |z[\varphi]|$. Then $(v, z[\varphi]) < 0$ and hence $\varphi(v) \le 0$, as required.

If m is sufficiently large $\rho_2(\alpha, n) + \delta < \sqrt{m} - 1$ and hence $|v + q' - y'| \le |v| + |q' - y'| \le \rho_2(\alpha, n) + \delta < \sqrt{m} - 1 < |q'' - y''|$. Then $\{v + q' - y', v + q'' - y''\} \prec \{q' - y', q'' - y''\}$, as required.

LEMMA 3.2. *There exists a sufficiently large positive integer ν, such that if $q', q'' \in Q$, $y', y'' \in \mathbb{R}^n$, $|q' - y'| \ge \nu$, $|q'' - y''| \ge \nu$, and $(q' - y')/|q' - y'|$, $(q'' - y'')/|q'' - y''| \in -\Sigma_A^c(Q)$ then there is an element $v \in -Q_\varphi = \{q \in Q | \varphi(q) \le 0\}$ with $|q' - y' + v| < |q' - y'|$, $|q'' - y'' + v| < |q'' - y''|$.*

Proof. If ν and m are sufficiently big positive integers we can apply Lemma 1.7 for the set $X = \{q' - y', q'' - y'', m[\varphi]\}$. Note that by Lemma 1.4 the set X satisfies the assumptions of Lemma 1.7. Then there exists $v \in Q$ such that $|v + q' - y'| < |q' - y'|$, $|v + q'' - y''| < |q'' - y''|$, $|v + m[\varphi]| < |m[\varphi]|$. The latter shows $\varphi(v) < 0$.

THEOREM 3.3. *There exists a positive real number $\rho^* \ge \rho$ such that if $m \ge \rho^*$, $q' \in O_{\sqrt{m}}(y') \subset X(m)$, $q'' \in O_{\sqrt{m}}(y'') \subset X(m)$, $q', q'' \in Q$, $y', y'' \in \mathbb{R}^n$ the commutator $[{}^{q'}c_i, {}^{q''}c_j]$ belongs to the subgroup of F/N generated by all F/N-conjugates of $\{{}^{Q_\varphi}[{}^{h'}c_i, {}^{h''}c_j] | h', h'' \in Q, \{h' - y', h'' - y''\} \prec \{q' - y', q'' - y''\}\}$, $\{{}^{Q_\varphi}c_{\alpha,\beta} | 1 \le \alpha \le s, 1 \le \beta \le m_0\}$ and $\{{}^{Q_\varphi}[{}^{h'}c_\alpha, {}^{h''}c_\beta] | 1 \le \alpha, \beta \le s; h', h'' \in X(m-1) \cap Q\}$.*

Proof. We assume $m \ge \rho$ and consider several cases. If $(q' - y')/|q' - y'| \in -\Sigma_A(Q)$ and $|q' - y'| > \rho_0$ we deduce from Lemma 2.2 that ${}^{q'}c_i$ belongs to the subgroup of F/N generated by $\{{}^q c_i | q \in Q, |q - y'| < |q' - y'|\} \cup \{{}^{Q_\varphi}c_{i,\beta} | 1 \le \beta \le m_0\}$ and all F/N-conjugates of $\{{}^{Q_\varphi}[{}^{h'}c_i, {}^{h''}c_i] | h', h'' \in X(m-1) \cap Q\}$. Then $[{}^{q'}c_i, {}^{q''}c_j]$ belongs to the normal subgroup of F/N generated by $\{[{}^q c_i, {}^{q''}c_j] | q \in Q, |q - y'| < |q' - y'|\}$, $\{{}^{Q_\varphi}c_{i,\beta} | 1 \le \beta \le m_0\}$, and $\{{}^{Q_\varphi}[{}^{h'}c_i, {}^{h''}c_i] | h', h'' \in X(m-1) \cap Q\}$, a subgroup of the normal subgroup of F/N described in the theorem.

The case when $(q'' - y'')/|q'' - y''| \in -\Sigma_A(Q)$ and $|q'' - y''| > \rho_0$ is the same as the one described above.

There are four possibilities left:

1. $|q' - y'| \le \rho_0$, $|q'' - y''| \le \rho_0$;
2. $|q' - y'| \le \rho_0$, $(q'' - y'')/|q'' - y''| \in -\Sigma_A^c(Q)$;
3. $|q'' - y''| \le \rho_0$, $(q' - y')/|q' - y'| \in -\Sigma_A^c(Q)$;
4. $(q' - y')/|q' - y'|$, $(q'' - y'')/|q'' - y''| \in -\Sigma_A^c(Q)$.

If 1 holds we prove that $q', q'' \in X(m-1)$ provided m is sufficiently big. Indeed

$$\left| q' - \frac{m-1-\sqrt{m-1}}{m-\sqrt{m}} y' \right| \le |q' - y'| + \left| y' - \frac{m-1-\sqrt{m-1}}{m-\sqrt{m}} y' \right|$$
$$\le \rho_0 + \left(\frac{1-\sqrt{m} + \sqrt{m-1}}{m-\sqrt{m}} \right) |y'|$$
$$\le \rho_0 + 1 - \sqrt{m} + \sqrt{m-1} \le \sqrt{m-1}$$

for sufficiently big m. We note that

$$\left| \frac{m-1-\sqrt{m-1}}{m-\sqrt{m}} y' \right| \le m - 1 - \sqrt{m-1}$$

and hence

$$q' \in O_{\sqrt{m-1}} \left(\frac{m-1-\sqrt{m-1}}{m-\sqrt{m}} y' \right) \subset X(m-1).$$

If 2 holds and m is sufficiently big $q' \in X(m-1)$. If $q'' \in X(m-1)$ we are done. Then we can assume $q'' \notin X(m-1)$. Let v be the element of $-Q_\varphi$ given by Lemma 3.1 for $\delta = \rho_0$ and $m \ge \mu_1(\delta)$; i.e., $\{v + q' - y', v + q'' - y''\} \prec \{q' - y', q'' - y''\}$. Then $[{}^{q'}c_i, {}^{q''}c_j] = {}^{v^{-1}}[{}^{vq'}c_i, {}^{vq''}c_j]$ belongs to the normal subgroup of F/N generated by $\{{}^{Q_\varphi}[{}^{h'}c_i, {}^{h''}c_j] \| h', h'' \in Q, \{h' - y', h'' - y''\} \prec \{q' - y', q'' - y''\}\}$. This completes case 2. Case 3 is the same as case 2.

Finally we are left to deal with case 4. We assume that $|q' - y'| \ge \nu$, $|q'' - y''| \ge \nu$, where ν is the positive integer given by Lemma 3.2, and let $v \in -Q_\varphi$ be the element given by Lemma 3.2. Then $[{}^{q'}c_i, {}^{q''}c_j] = {}^{v^{-1}}[{}^{vq'}c_i, {}^{vq''}c_j]$ belongs to the normal subgroup of F/N generated by $\{{}^{Q_\varphi}[{}^{h'}c_i, {}^{h''}c_j] \| h', h'' \in Q, \{h' - y', h'' - y''\} \prec \{q' - y', q'' - y''\}\}$. To finish the proof we have to consider the case when $(q' - y')/|q' - y'|, (q'' - y'')/|q'' - y''| \in -\Sigma_A^c(Q)$ and $|q' - y'| < \nu$ or $|q'' - y''| < \nu$. Here we have to repeat the arguments of case 2 and case 3 with ν instead of ρ_0. This completes the proof of the theorem.

COROLLARY 3.4. *For $m \ge \rho^*$ and $q', q' \in X(m) \cap Q$ the commutator $[{}^{q'}c_i, {}^{q''}c_j]$ belongs to the normal subgroup of F/N generated by all F/N-conjugates of $\{{}^{Q_\varphi}[{}^{h'}c_\alpha, {}^{h''}c_\beta] \| 1 \le \alpha, \beta \le s;\ h', h'' \in X(m-1) \cap Q\}$ and $\{{}^{Q_\varphi}c_{\alpha,\beta} | 1 \le \alpha \le s, 1 \le \beta \le m_0\}$.*

COROLLARY 3.5. *In the assumptions of Section* 1.2 Ker μ *is generated by finitely many* Q_φ*-orbits as a normal subgroup of* F.

Proof. By Corollary 3.4 if we quotient F through the normal subgroup of F generated by N, $\{{}^{Q_\varphi}(b_i)[({}^{z_{j,1}q_{j,1}}b_i)({}^{z_{j,2}q_{j,2}}b_i)\cdots({}^{z_{j,t_j}q_{j,t_j}}b_i)]^{-1})|1 \le i \le s, 1 \le j \le m_0\}$ and $\{{}^{Q_\varphi}[{}^{h'}b_i, {}^{h''}b_j]|1 \le i, j \le s;\ h', h'' \in X(\rho^* - 1) \cap Q\}$ we obtain an abelian group A_0 equipped with a left Q_φ-action such that the kernel of the map $F \to A_0$ is in Ker μ. By Corollary 3.4 and Lemma 1.2 we see that A_0 is equipped with Q-action inherited from the Q-action on F/N. Then A is a surjective image of A_0 as a left $Z\mathbb{Q}$-module and hence the kernel of the projection map from A_0 to A is finitely generated as a $\mathbb{Z}Q$-module. Using Lemma 1.2 we see that this kernel is a finitely generated $\mathbb{Z}Q_\varphi$-module as well.

4. PROOF OF THEOREM A1

In this section we prove Theorem A1 by constructing a 1-connected 2-complex X on which G acts freely and cocompactly. An obvious choice of a CW-complex acted on by G is the Cayley graph Γ of A with respect to some free Q-set of finite rank which maps surjectively to a generating set of A. Now two problems arise: Γ is not 1-connected and since G acts on Γ with non-trivial vertex stabilizers it is impossible to define a χ-equivariant height function on Γ. It is easy to overcome the first obstacle because since G is finitely presented Γ can be embedded in a 1-connected 2-complex $\tilde{\Gamma}$ on which G acts cocompactly.

There is a general construction (see [19, Theorem 3.1]) that for every finitely generated group G, not necessarily metabelian, and for every connected 2-complex on which G acts cocompactly with finitely presented vertex stabilizers and finitely generated edge stabilizers, "blows up" this complex to a 1-connected 2-complex on which G acts freely and cocompactly. Our complex X is obtained by "blowing up" $\tilde{\Gamma}$ and removing some 2-cells with boundaries representing combinatorially trivial paths. We do not assume that the reader is familiar with Meinert's construction [19, Theorem 3.1] and describe X in detail as this will be necessary for the proof of Theorem 4.3.

Let G be a group satisfying the assumptions of Theorem A1. Since G is a metabelian group of type FP_2 by [8] A is 2-tame as a $\mathbb{Z}Q$-module; i.e., $\Sigma_A^c(Q) \cap -\Sigma_A^c(Q) = \emptyset$ and we can use the results from the previous sections. Without loss of generality we assume that Q is a free abelian group of rank n with a basis $e_1, \dots, e_n$.

We consider A as a (left) $\mathbb{Z}Q$-module via left conjugation and fix a generating set $a_1, \dots, a_s$ of A as a $\mathbb{Z}Q$-module. Now we define 2-complex

Y such that its 1-skeleton $Y^{(1)}$ is the Cayley graph of G with respect to the generating system $\mathscr{B} = \{a_1, \ldots, a_s, e_1, \ldots, e_n\}$; i.e., the vertices of $Y^{(1)}$ are the elements of G, the edges are $(g, b) \in G \times \mathscr{B}$, with vertices g and gb, and G acts via left multiplications. Let $Y_a^{(1)}$, $a \in A$ denote the full subcomplex of $Y^{(1)}$ spanned by the vertices aQ. Note that $Y_a^{(1)}$ is the 1-skeleton $\bigcup_{0 \le i \le n-1} \mathbb{Z}^i \times \mathbb{R} \times \mathbb{Z}^{n-i-1}$ of $\mathbb{R}^n$. Finally Y is defined to be the 2-complex built by gluing copies of the 2-skeleton $\bigcup_{0 \le i_1 + i_2 \le n-2} \mathbb{Z}^{i_1} \times \mathbb{R} \times \mathbb{Z}^{i_2} \times \mathbb{R} \times \mathbb{Z}^{n-i_1-i_2-2}$ of $\mathbb{R}^n$ to every $Y_a^{(1)}, a \in A$. The full subcomplex of Y spanned by the vertices aQ for some $a \in A$ is denoted Y_a.

LEMMA 4.1. *Let Z be the 1-complex obtained from Y by squeezing all subcomplexes Y_a, $a \in A$ to different points and p: $Y \to Z$ be the corresponding projection map. Then Y and Z are both connected and the induced map p_*: $\pi_1(Y) \to \pi_1(Z)$ is an isomorphism. The action of G on Y induces via p an action of G on Z.*

Proof. The lemma follows immediately from the fact that Y_a is simply connected for all $a \in A$.

We note that Z is the Cayley graph of the abelian group A with respect to the map $Q \times \{1, \ldots, s\} \to \{{}^q a_i | q \in Q,\ 1 \le i \le s\}$ sending (q, i) to ${}^q a_i$; i.e., the set of vertices is A and a typical edge $(a, q, i) \in A \times Q \times \{1, \ldots, s\}$ has vertices a and $a + {}^q a_i$. The map p sends the edge $(g, a_i) \in G \times \mathscr{B}$ to $(a, q, i) \in A \times Q \times \{1, \ldots, s\}$ and the vertex $g \in Y$ to the vertex $a \in Z$, where $g = aq$ in G. Note that the action of G on Z induced by the map p is the following: A acts on the set of vertices of Z via left multiplication and Q via (left) conjugation on A.

It is easy to see that $\pi_1(Z) \simeq R$, where R is the kernel of the surjective homomorphism from the free group F on the set $Q \times \{1, \ldots, s\}$ to A sending (q, i) to ${}^q a_i$. Since $G \simeq (F \rtimes Q)/R$ is finitely presented R is finitely generated as a normal subgroup of $(F \rtimes Q)$, say by $r_1, \ldots, r_m$. Now we attach a free G-set $\{{}^G\widetilde{c_1}, {}^G\widetilde{c_2}, \ldots, {}^G\widetilde{c_m}\}$ of 2-cells to Z, where the boundary of $\widetilde{c_i}$ represents the element r_i of $\pi_1(Z)$ and denote by Z_0 this new 2-complex. By construction Z_0 is 1-connected.

DEFINITION. We attach a free G-set of 2-cells $\{{}^G c_1, {}^G c_2, \ldots, {}^G c_m\}$ to Y such that the boundary of ${}^g c_i$ under the map p coincides with the boundary of ${}^g \widetilde{c_i}$ for all $g \in G$, $1 \le i \le m$ and denote by X this new 2-complex. Let p: $X \to Z_0$ be a continuous G-map lifting the map p described in Lemma 4.1 with the additional property that p sends c_i to $\widetilde{c_i}$ for all $1 \le i \le m$.

The following lemma is an immediate consequence of the construction of X and Lemma 4.1.

LEMMA 4.2. *X is a 1-connected 2-complex on which G acts freely and cocompactly. The vertex set of X is the underlying set of the group G.*

Since G acts freely on X it is easy to construct a χ-equivariant height function $h = h_\chi\colon X \to \mathbb{R}$, where $\chi = \gamma(\varphi) \in \operatorname{Hom}(G, \mathbb{R})$. For a vertex $g \in G$ we set $h(g) := \chi(g)$ and extend h to a regular χ-equivariant height function on X; i.e., h is a continuous function with $h(gx) = h(x) + \chi(g)$ for every $g \in G$, $x \in X$, such that the restriction of h to a cell of X always attains its extreme values on the cell boundary.

The main step of the proof of Theorem A1 is Theorem 4.3. It uses substantially the main result of Section 3, i.e., Corollary 3.5. Once Theorem 4.3 is proved Meinert's criterion [19, Theorem 4.1] completes the proof of Theorem A1.

THEOREM 4.3. *$X_h^{[0,+\infty)}$ is essentially 1-connected in X.*

Proof. We remind the reader that by definition $X_h^{[r,+\infty)}$ is the maximal subcomplex of X contained in $h^{-1}([r,+\infty))$. We note that $X = \bigcup_{r \in \mathbb{R}} X_h^{[r,+\infty)}$ and since $\chi \in \Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$ all $\{X_h^{[r,+\infty)}\}_{r \in \mathbb{R}}$ are connected. Similarly to Lemma 4.1 we obtain

$$\pi_1\big(X_h^{[r,+\infty)}\big) \simeq \pi_1\big(Z_{0,h}^{[r,+\infty)}\big),$$

where $Z_{0,h}^{[r,+\infty)} = p(X_h^{[r,+\infty)})$. Let $Z_h^{[r,+\infty)}$ denote $p(Y \cap X_h^{[r,+\infty)})$. Then $Z_{0,h}^{[r,\infty)}$ is obtained from $Z_h^{[r,\infty)}$ by gluing those 2-cells ${}^g\widetilde{c_i}$ for which the minimal value of h on ${}^g c_i$ is bounded below by r; i.e., h (any vertex of $c_i) \geq r - \chi(g)$. Note that the vertices of $Z_h^{[r,\infty)}$ are the elements of A and the set of edges of $Z_h^{[r,\infty)}$ is $A \times \{q \in Q \mid \varphi(q) \geq r\} \times \{1, 2, \dots, s\}$.

Let F_r be the free group on the set $Y_r = \{q \in Q \mid \varphi(q) \geq r\} \times \{1, 2, \dots, s\}$, where $r \leq 0$, and so F_{r_1} is a subgroup of F_{r_2} for $r_1 \geq r_2$. We denote by R_r the kernel of the surjective group homomorphism from F_r to A sending $(q, i) \in Y_r$ to ${}^q a_i \in A$. Then

$$\pi_1\big(Z_h^{[r,\infty)}, v_0\big) \simeq R_r \qquad \text{for all } r \leq 0,$$

where the isomorphism is induced by the map that sends every closed combinatorial path in $Z_h^{[r,\infty)}$ attached at the vertex $v_0 = 1_A$ to its label. The label of the edge $(a, q, i) \in A \times \{q \in Q \mid \varphi(q) \geq r\} \times \{1, \dots, s\}$ is ${}^q a_i$ and the label of a path is the product of the labels of the edges in the order they appear in the path. Note that $F_r \rtimes Q_\varphi$ acts on R_r, F_r via (left) conjugation and Q_φ via its (left) action on Y_r given by (left) multiplication. By Corollary 3.5 R_0 is finitely generated as a (left) $F_0 \rtimes Q_\varphi$-group, say by

$\alpha_1, \dots, \alpha_t$. Since

$$\lim_{j \to -\infty} \pi_1\big(Z_{0,h}^{[j,\infty)}, v_0\big) = \pi_1(Z_0, v_0) = 0$$

there exists a negative integer j_0 such that the images of $\alpha_1, \dots, \alpha_t$ in $\pi_1(Z_{0,h}^{[j_0,\infty)}, v_0)$ are trivial. Now we consider the commutative diagram

$$\begin{array}{ccc} R_0 \simeq \pi_1(Z_h^{[0,\infty)}, v_0) & \xrightarrow[i]{} & \pi_1(Z_h^{[j_0,\infty)}, v_0) \simeq R_{j_0} \\ \big\downarrow \mu_0 & & \big\downarrow \mu_{j_0} \\ \pi_1(Z_{0,h}^{[0,\infty)}, v_0) & \xrightarrow[i_0]{} & \pi_1(Z_{0,h}^{[j_0,\infty)}, v_0), \end{array} \qquad (*)$$

where all the maps are induced by the obvious inclusions of spaces. We view $\pi_1(Z_{0,h}^{[0,\infty)}, v_0)$ and $\pi_1(Z_{0,h}^{[j_0,\infty)}, v_0)$ as $F_0 \rtimes Q_\varphi$-groups via the maps μ_0 and μ_{j_0}, respectively. Then the maps of the diagram $(*)$ commute with the action of $F_0 \rtimes Q_\varphi$. By the choice of j_0, $\mu_{j_0} i = i_0 \mu_0 = 0$ and since μ_0 is surjective i_0 is the trivial map. Then the map $\pi_1(X_h^{[0,\infty)}, v_0) \to \pi_1(X_h^{[j_0,\infty)}, v_0)$ induced by the inclusion of the corresponding spaces is trivial; i.e., $X_h^{[0,\infty)}$ is essentially 1-connected as required,

5. PROOF OF THEOREM B

In this section we assume again that G is a split extension of A by Q, A and Q are abelian, and G is finitely generated. We consider B a finitely generated $\mathbb{Z}Q$-module and view B as a module over $\mathbb{Z}G$ via the projection $G \to Q$. The following lemma is a starting point of our proof of Theorem B.

LEMMA 5.1. *If Q is a finitely generated abelian group and B is a finitely generated $\mathbb{Z}Q$-module then $\Sigma^m(Q, B)^c = \Sigma^0(Q, B)^c$ for all $m \in \mathbb{N}$.*

Proof. We choose a free resolution $\mathscr{C}$ of B whose components C_i are free $\mathbb{Z}Q$-modules of finite rank. In addition we assume that the resolution is admissible in Bieri–Renz sense as defined in [7, p. 471]; i.e., for every $i \geq 0$ the free $\mathbb{Z}Q$-module C_i is endowed with a basis X_i such that the differential does not send any element of X_i to zero. We aim to show that if χ is a non-trivial real character of Q such that $[\chi] \in \Sigma^0(Q, B)$ then $[\chi] \in \Sigma^m(Q, B)$. By [8, Proposition 2.1] there exists λ in the centralizer of B in $\mathbb{Z}Q$ such that for all $q \in \operatorname{supp} \lambda$, $\chi(q) > 0$. Then we define a chain map $\nu\colon \mathscr{C} \to \mathscr{C}$ such that in non-negative dimensions ν is multiplication by λ and in dimension -1 is the identity of B. Let v be the Bieri–Renz valuation on $\mathscr{C}$ associated to the character χ; i.e., the restriction v_i of v to the free module C_i is a map to $\mathbb{R} \cup \{\infty\}$ satisfying the following axioms:

1. $v_i(0) = \infty$;
2. $v_i(gx) = \chi(g) + v_i(x)$ for $g \in G$, $x \in X_i$;
3. $v_i(f) = \min\{v_i(y) | n_y \neq 0\}$ if $f = \sum n_y y \neq 0$ is the unique expression of $f \in C_i$ in terms of the $\mathbb{Z}$-basis GX_i, $n_y \in \mathbb{Z}$;
4. the values of v_i on X_i are chosen inductively by $v_0(x) = 0$ and $v_i(x) = v_{i-1}(\partial x)$ for $i > 0$.

Then the choice of λ implies $v(\nu(x)) > v(x)$ for all $x \in \bigcup_{i \geq 0} X_i$.

Finally the Bieri–Renz criterion [7, Theorem 4.1] implies $[\chi] \in \Sigma^m(Q, B)$ for all $m \in \mathbb{N}$.

LEMMA 5.2. *If* $\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\}$, $\chi(A) = 0$, *and* $[\chi] \in \Sigma^0(G, B)$ *then B is not finitely presented over $\mathbb{Z}G_\chi$ if and only if $B \otimes \operatorname{Aug}\mathbb{Z}A$ is not finitely generated over $\mathbb{Z}G_\chi$ via the diagonal action. B is finitely presented over $\mathbb{Z}G$ if and only if $B \otimes \operatorname{Aug}\mathbb{Z}A$ is finitely generated over $\mathbb{Z}G$ (via the diagonal G-action).*

Proof. By Lemma 5.1 B is of type FP_∞ over $\mathbb{Z}Q_\chi$. We consider the short exact sequence of $\mathbb{Z}G_\chi$-modules

$$0 \to \operatorname{Aug}\mathbb{Z}A \otimes_{\mathbb{Z}} B \to \mathbb{Z}A \otimes_{\mathbb{Z}} B \simeq \mathbb{Z}G_\chi \otimes_{\mathbb{Z}Q_\chi} B \to B \to 0$$

given by inducing the short exact sequence $0 \to \operatorname{Aug}\mathbb{Z}A \to \mathbb{Z}A \to \mathbb{Z} \to 0$. Now the first part of the lemma follows from [2, Proposition 1.4] and the fact that $\mathbb{Z}G_\chi \otimes_{\mathbb{Z}Q_\chi} B$ is induced from a module of type FP_∞ over $\mathbb{Z}Q_\chi$ and hence is of type FP_∞ over $\mathbb{Z}G_\chi$. Similarly since $\mathbb{Z}A \otimes_{\mathbb{Z}} B \simeq \mathbb{Z}G \otimes_{\mathbb{Z}Q} B$ we see that $\operatorname{Aug}\mathbb{Z}A \otimes_{\mathbb{Z}} B$ is finitely generated over $\mathbb{Z}G$ (via the diagonal G-action) if and only if B is finitely presented over $\mathbb{Z}G$.

PROPOSITION 5.3. *If B is finitely presented over $\mathbb{Z}G$*

$$\mathbb{R}_{>0}\Sigma^0(G, A)^c + \mathbb{R}_{>0}\Sigma^0(G, B)^c \subseteq \mathbb{R}_{>0}\Sigma^1(G, B)^c.$$

Remark. $\Sigma^0(G, A)^c = \Sigma^1(G)^c$.

Proof. The proof generalizes ideas from [15, Proposition 6.3]. We assume there exist non-trivial real characters χ_1, χ_2 of Q such that $[\chi_1] \in \Sigma_A^c(Q) \simeq \Sigma^0(G, A)^c$, $[\chi_2] \in \Sigma_B^c(Q) \simeq \Sigma^0(G, B)^c$ and $\chi_1 + \chi_2 = \chi_0$ is either 0 or $[\chi] \in \Sigma^1(G, B)$ for $\chi = \chi_0 \circ \pi$ where π is the projection from G to Q and the above isomorphisms between geometric invariants are induced from the projection π. By Lemma 5.2 $\operatorname{Aug}\mathbb{Z}A \otimes B$ is finitely generated over $\mathbb{Z}G_\chi$. Since $[\chi_1] \in \Sigma_A^c(Q)$ and $[\chi_2] \in \Sigma_B^c(Q)$ there are filtrations $v_1\colon A \to \mathbb{R}_\infty$, $v_2\colon B \to \mathbb{R}_\infty$ such that $v_1({}^q a) = \chi_1(q) + v_1(a)$,

$v_1(a_1 + a_2) \geq \min\{v_1(a_1), v_1(a_2)\}$, $\operatorname{Im} v_1 \neq \{\infty\}$ and $v_2({}^q b) = \chi_2(q) + v_2(b)$, $v_2(b_1 + b_2) \geq \min\{v_2(b_1), v_2(b_2)\}$, $\operatorname{Im} v_2 \neq \{\infty\}$. Now we define $\rho\colon \operatorname{Aug}\mathbb{Z}A \otimes B \to \mathbb{R}_\infty$ by

$$\rho(\lambda) = \sup\Big\{d \mid \lambda \in \Sigma_{d_1 + d_2 \geq d,\, d_i \in \operatorname{Im} \chi_i} \operatorname{Ker}\Big(\mathbb{Z}A \to \mathbb{Z}\big(A/v_1^{-1}([d_1, \infty))\big)\Big) \otimes v_2^{-1}([d_2, \infty))\Big\}.$$

As shown in [15, Proposition 6.3] ρ satisfies the following properties

1. $\rho((a-1)\otimes b) = v_1(a) + v_2(b)$;
2. $\rho({}^g(a-1)\otimes^g b) = v_1({}^g a) + v_2({}^g b) = \chi(g) + \rho((a-1)\otimes b)$;
3. $\rho(\lambda_1 + \lambda_2) \geq \min\{\rho(\lambda_1), \rho(\lambda_2)\}$.

The second and the third properties together with the fact that $\operatorname{Aug}\mathbb{Z}A \otimes B$ is finitely generated over $\mathbb{Z}G_\chi$ imply that $\operatorname{Im}\rho$ is bounded below, a contradiction with the first property of ρ.

THEOREM 5.4. *If B is finitely presented over $\mathbb{Z}G$ then*

$$\mathbb{R}_{>0}\Sigma^1(G, B)^c \subseteq \mathbb{R}_{>0}\Sigma^0(G, B)^c + \mathbb{R}_{>0}\Sigma^0(G, A)^c.$$

Proof. We assume that χ is a non-trivial real character of G such that $\chi \notin \mathbb{R}_{>0}\Sigma^0(G, B)^c + \mathbb{R}_{>0}\Sigma^0(G, A)^c$ and aim to prove that $[\chi] \in \Sigma^1(G, B)$. As shown in [17] the real characters of G that do not vanish on A represent elements of $\Sigma^1(G, B)$ and therefore we can assume that $\chi(A) = 0$; i.e., $\chi = \varphi \circ \pi$ for some non-trivial real character φ of Q. By Lemma 5.2 it suffices to prove that $B \otimes \operatorname{Aug}\mathbb{Z}A$ is finitely generated over $\mathbb{Z}G_\chi$. Our proof of Theorem 5.4 uses the geometric methods developed in the proof of Theorem A1 but it is easier as there is already an action of Q on A and B. Note that in the assumption of Theorem B the $\mathbb{Z}Q$-module A might not be 2-tame. The role of 2-tameness will be played by the condition $\Sigma_A^c(Q) \cap -\Sigma_B^c(Q) = \emptyset$. The latter holds because as shown in [16, Proposition 3] it is equivalent with B being finitely presented over $\mathbb{Z}G$. Now we state several lemmas that are obvious modifications of the geometric lemmas included in the previous sections.

LEMMA 5.4.1. (See 1.3.) *If $\Sigma_A^c(Q) \cap -\Sigma_B^c(Q) = \emptyset$, $x_1 \in -\Sigma_A^c(Q)$ and $x_2 \in -\Sigma_B^c(Q)$ the angle between x_1 and x_2 cannot be arbitrary close to π.*

LEMMA 5.4.2. (See 1.4.) *If $\varphi \notin \mathbb{R}_{>0}\Sigma_B^c(Q) + \mathbb{R}_{>0}\Sigma_A^c(Q)$ there exists a positive real number $\alpha \in (0, \pi)$ such that for every set $X = \{x_1, x_2, x_3\} \subset \mathbb{R}^n \simeq \operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$, $x_1 \in -\Sigma_A^c(Q)$, $x_2 \in -\Sigma_B^c(Q)$, and $x_3 \in \mathbb{R}_{>0}[\varphi]$ there exists $u \in \mathbb{R}^n$ such that $\angle\{x, u\} < \frac{\pi}{2} - \frac{\alpha}{2}$ for all $x \in X$.*

To prove Theorem 5.4 we use the main idea of the proof of Corollary 3.4 with some modifications given by the following lemmas. Most of the proofs are omitted as they are the same as the proofs of the corresponding statements of Sections 1.2, 2, and 3. Note that we do not need analogue statements of all geometric results from Sections 1.2, 2, and 3, as there is already an action of Q on A and B and A and B are abelian (we should not worry about expressions of $c_{i,j}$'s as in the proof of Theorem A1). For example we do not need an analogue of Lemma 2.1, where a special positive integer ρ is defined.

LEMMA 5.4.3. (See 1.5 and 1.6.) *There exist a finite subset Λ of $C_{\mathbb{Z}Q}(A) \cup C_{\mathbb{Z}Q}(B)$ and a positive real number ρ_0 such that for $|x| > \rho_0$, $\frac{x}{|x|} \in -\Sigma_M(Q)$, $M = A$ or B, there exists $\lambda \in \Lambda \cap C_{\mathbb{Z}Q}(M)$ with the property that for every $q \in \operatorname{supp} \lambda$, $|x + q| < |x|$.*

LEMMA 5.4.4.1. (See 2.2.) *Let $\mathscr{X}_B$ be a finite generating set of B over $\mathbb{Z}Q$. Suppose $q' \in O_{\sqrt{m}}(y') \subset X(m)$, $q' \in Q$, $y' \in \mathbb{R}^n$, and $|q' - y'| > \rho_0$, $(q' - y')/|q' - y'| \in -\Sigma_B(Q)$. Then for $c \in \mathscr{X}_B$ the element ${}^{q'}c$ belongs to the subgroup of B generated by $\{{}^{q}c \mid q \in Q, |q - y'| < |q' - y'|\}$.*

LEMMA 5.4.4.2. *Let $\mathscr{X}_A$ be a finite generating set of A over $\mathbb{Z}Q$. Suppose $q' \in O_{\sqrt{m}}(y') \subset X(m)$, $q' \in Q$, $y' \in \mathbb{R}^n$, and $|q' - y'| > \rho_0$, $(q' - y')/|q' - y'| \in -\Sigma_A(Q)$. Then for $c \in \mathscr{X}_A$ the element ${}^{q'}(c - 1)$ belongs to the $\mathbb{Z}A$-submodule of the augmentation ideal of $\mathbb{Z}A$ generated by $\{{}^{q}(c - 1) \mid q \in Q, |q - y'| < |q' - y'|\}$.*

Note that Lemma 5.4.4.2 follows from Lemma 5.4.4.1 applied for the $\mathbb{Z}Q$-module A.

LEMMA 5.4.5. (See 3.1.) *Suppose δ is a real positive number. Then there exists a positive real number $\mu_1(\delta)$, depending on δ, such that if $m \geq \mu_1(\delta)$, $q', q'' \in Q$, $y', y'' \in \mathbb{R}^n$, $q' \in O_{\sqrt{m}}(y') \subset X(m)$, $q'' \in O_{\sqrt{m}}(y'') \subset X(m)$, and $|q' - y'| \leq \delta$, $q'' \notin X(m - 1)$, $(q'' - y'')/|q'' - y''| \in -\Sigma_A^c(Q) \cup -\Sigma_B^c(Q)$ then there exists $v \in -Q_\varphi$ such that $\{v + q' - y', v + q'' - y''\} \prec \{q' - y', q'' - y''\}$.*

LEMMA 5.4.6. (See 3.2.) *There exists a sufficiently large positive integer ν such that if $q', q'' \in Q$, $y', y'' \in \mathbb{R}^n$, $|q' - y'| \geq \nu$, $|q'' - y''| \geq \nu$, and $(q' - y')/|q' - y'| \in -\Sigma_B^c(Q)$, $(q'' - y'')/|q'' - y''| \in -\Sigma_A^c(Q)$ then there is an element $v \in -Q_\varphi = \{q \in Q \mid \varphi(q) \leq 0\}$ with $|q' - y' + v| < |q' - y'|$, $|q'' - y'' + v| < |q'' - y''|$.*

THEOREM 5.4.7. (See 3.3.) *As before let $\mathscr{X}_A$ and $\mathscr{X}_B$ be finite generating sets of A and B, respectively, over $\mathbb{Z}Q$. Then there exists a positive real number ρ^* such that if $m \geq \rho^*$, $q' \in O_{\sqrt{m}}(y') \subset X(m)$, $q'' \in O_{\sqrt{m}}(y'') \subset$*

$X(m)$, $q', q'' \in Q$, and $y', y'' \in \mathbb{R}^n$ then for $b \in \mathscr{X}_B$, $a \in \mathscr{X}_A$ the element ${}^{q'}b \otimes^{q''} (a-1)$ belongs to the $\mathbb{Z}G_\chi$-submodule of $B \otimes \operatorname{Aug} \mathbb{Z}A$ generated by $\{{}^{h'}b \otimes^{h''} (a-1) | b \in \mathscr{X}_B,\ a \in \mathscr{X}_A,\ h', h'' \in Q,\ \{h'-y', h''-y''\} \prec \{q'-y', q''-y''\}\}$ and $\{{}^{h'}b \otimes^{h''} (a-1) | a \in \mathscr{X}_A, b \in \mathscr{X}_B, h', h'' \in X(m-1) \cap Q\}$.

Proof. The proof of Theorem 5.4.7 follows from the modified lemmas in the same way as Theorem 3.3 follows from the geometric lemmas in Sections 1.2, 2, and 3. We omit the details and only state the cases that should be considered.

1. $(q'-y')/|q'-y'| \in -\Sigma_B(Q)$ and $|q'-y'| > \rho_0$ (use Lemma 5.4.4.1);
2. $(q''-y'')/|q''-y''| \in -\Sigma_A(Q)$ and $|q''-y''| > \rho_0$ (use Lemma 5.4.4.2);
3. $|q'-y'| \le \rho_0$, $|q''-y''| \le \rho_0$ (in this case $q', q'' \in X(m-1)$ for sufficiently big m);
4. $|q'-y'| \le \rho_0$, $(q''-y'')/|q''-y''| \in -\Sigma_A^c(Q)$ (use Lemma 5.4.5);
5. $|q''-y''| \le \rho_0$, $(q'-y')/|q'-y'| \in -\Sigma_B^c(Q)$ (proceed as in the previous case);
6. $(q'-y')/|q'-y'| \in -\Sigma_B^c(Q), (q''-y'')/|q''-y''| \in -\Sigma_A^c(Q)$ (use Lemma 5.4.6 to reduce to the case when $|q''-y''| \le \nu$ or $|q'-y'| \le \nu$. Then use the idea of case 5 or 4, respectively).

The following corollary completes the proof of Theorem 5.4.

COROLLARY 5.4.8. (See 3.4.) *If $m \ge \rho^*$, $q', q'' \in X(m) \cap Q$ then for $a \in \mathscr{X}_A$, $b \in \mathscr{X}_B$, ${}^{q'}b \otimes^{q'} (a-1)$ belongs to the $\mathbb{Z}G_\chi$-submodule of $B \otimes \operatorname{Aug} \mathbb{Z}A$ generated by $\{{}^{h'}b \otimes^{h''} (a-1) | b \in \mathscr{X}_B,\ a \in \mathscr{X}_A,\ h', h'' \in X(m-1) \cap Q\}$. In particular $B \otimes \operatorname{Aug} \mathbb{Z}A$ is finitely generated over $\mathbb{Z}G_\chi$.*

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